5 Evaluating expressions

In this first topic, we will look at evaluating expressions, including the evaluation of

1. a polynomial at a point,
2. a derivative at a point, and
3. definite integrals.

In each case, we will see that there are issues, some of which may be mitigated, and in others will have simply be identified and avoided. The most significant issue that will affect all of our evaluations is noise, and to deal with this, we will first use interpolation, but we will then look at least squares.

Noise

Engineering is about responding to real-world situations, where data is collected from sensors and analyzed, after which an appropriate response is determined. The data coming in from sensors, however, is never ideal and is always subject to additive noise.

One type of noise that is ubiquitous is *white noise*. We will consider white noise as described by its effect on a periodic reading taken by a sensor.

Suppose a sensor is reading temperature, the direction of a gyroscope or a GPS position. In each case, there is a true or correct reading, and assume that these correct readings are represented by the sequence

$$y_0, y_1, y_2, y_3, \cdots .$$

At each step, however, weaknesses in the sensor will introduce errors into each reading:

$$y_0 + \varepsilon_0, y_1 + \varepsilon_1, y_2 + \varepsilon_2, y_3 + \varepsilon_3, \cdots .$$

If the errors are independent of each other; that is, if the error ε_k gives you no additional information about what the error ε_{k+1}, then the errors have the additional property that it combines all possible frequencies, and hence given the term *white* noise. In some cases, it can be determined that the noise appears to be sampled from a Gaussian distribution (a bell curve, or normal distribution) with mean μ and standard deviation σ, in which case it is referred to as *Gaussian* white noise.

If the noise has a mean equal to zero, we say that the error is *bias-free*, while if the mean is non-zero, the noise is said to introduce a *bias* into our readings. For example, a heat source near a temperature sensor may always cause readings to be higher than expected. Such a heat source may be something as simple as a dark light-absorbing surface.

To determine if a sensor has a bias, it is necessary to test it under controlled conditions where the sensor is reading a known value, and then contrasting the sensor reading with the expected value. Suppose we read 20 or more such samples, and determine the errors

$$\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_n .$$

In this case, we calculate the sample average and sample standard deviation

$$\bar{\varepsilon} = \frac{1}{n+1}\sum_{k=0}^{n} \varepsilon_k$$

and

$$s = \sqrt{\frac{1}{n}\sum_{k=0}^{n} (\varepsilon_k - \bar{\varepsilon})^2} \, .$$

You may note that the second uses $\frac{1}{n}$ and not $\frac{1}{n+1}$; however there are excellent statistical reasons for this, specifically, because we are using the approximation $\bar{\varepsilon}$ in our second calculation, so we err on the side of caution, by over estimating. This is further explained in any introductory course in mathematical statistics.

To test whether or not the errors have a bias, we consider the value

$$\frac{\bar{\varepsilon}}{s/\sqrt{n}} \, .$$

If this exceeds 1.96, we can be 95 % confident that there is a bias in the noise, in which case, we can take steps to remove that bias from our sensor. By default, we assume that there is zero noise, and we only accept that the noise is biased if we are reasonably certain that there is a bias.

From this point, we will assume that any bias in a sensor has been removed and that any remaining noise is unbiased.

Summing noise versus differentiating unbiased noise

Let us consider the effect of unbiased noise on summing and differentiating a signal:

$$Y_n = \sum_{k=0}^{n} (y_k + \varepsilon_k) \, ,$$

then this equals

$$Y_n = \left(\sum_{k=0}^{n} y_k\right) + \left(\sum_{k=0}^{n} \varepsilon_k\right) .$$

If the noise is unbiased, the errors will on average cancel each other out, as $\sum_{k=0}^{n} \varepsilon_k \approx (n+1)\cdot 0 = 0$ will be approximately zero, so as *n* gets larger, so Y_n is a good approximation to $\sum_{k=0}^{n} y_k$. While the standard deviation may be $\sqrt{n+1}\cdot s$, this is likely small relative to Y_n. Additionally, if we defined

$$\bar{Y}_n = \frac{1}{n+1}\sum_{k=0}^{n} (y_k + \varepsilon_k) \, ,$$

the standard deviation now shrinks as *n* becomes larger: $\frac{s}{\sqrt{n+1}}$.

If we take the difference of two successive values, we have

$$\Delta_n = (y_{n+1} + \varepsilon_{n+1}) - (y_n + \varepsilon_n) = (y_{n+1} - y_n) + (\varepsilon_{n+1} - \varepsilon_n),$$

it can be shown that the standard deviation of Δ_n is now $\sqrt{2}s$, or larger. In calculating derivatives, however, we generally divide by a small value, so now the standard deviation is magnified: $\frac{\sqrt{2}s}{h}$. Thus, differentiation tends to amplify noise while integration attenuates it.

In conclusion, averaging signals with unbiased noise reduces the effect of noise, while differentiating signals with unbiased noise increases the effect.